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Tail Elasticities and Threshold-Dependent Aggregation*

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Abstract: This paper studies aggregation in economic models in which agents' decisions are governed by endogenous thresholds. The location of a threshold and the distribution of characteristics relative to it jointly determine aggregate outcomes. Examples include heterogeneous-firm models such as Melitz (2003) and financial accelerator models following Bernanke, Gertler, and Gilchrist (1999). We identify the generalized failure rate, which measures the elasticity of the upper-tail probability with respect to the cutoff, as the key selection statistic. We show that an increasing generalized failure rate is equivalent to higher cutoffs making the relative tail distribution smaller in the first-order stochastic order. We then define log-log concavity of the density, a condition equivalent to decreasing density elasticity, and show that it implies an increasing generalized failure rate. Building on this structure, we derive general monotonicity results for ratios of weighted upper-tail integrals, a class of objects that includes many aggregate distortions and equilibrium wedges. The framework yields tractable comparative statics for average productivity, aggregate markups, market concentration, trade elasticities, equilibrium default, and conditional tail risk. It also clarifies the special role of Pareto tails, for which relative tail distributions are invariant to the selection threshold and aggregate elasticities are constant.

JEL Classification: D30, D43, E44, F12, G33

Keywords: Endogenous selection, Thresholds, Tail elasticity, Log-log concavity, Trade elasticity, Equilibrium default, Aggregate distortion

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1 Introduction

In many economic models, participation, activity, and default decisions depend on the realization of idiosyncratic shocks. For example, firms enter export markets only if their productivity is sufficiently high (Melitz, 2003), borrowers repay only when project returns are sufficiently high (Bernanke et al., 1999), and searching workers accept jobs only if wages exceed a reservation level (McCall, 1970). Although these examples differ in their economic mechanisms, they share a common structure: aggregate variables depend on the distribution of characteristics relative to an endogenous threshold. The location of this threshold, together with the distribution of heterogeneity around it, is therefore central to aggregate outcomes. The question we address is when changes in such thresholds generate systematic and monotone aggregate responses.

The key objects we focus on are tail elasticities. In particular, the generalized failure rate measures the proportional sensitivity of the upper-tail probability to changes in the threshold. We show that monotonicity of this tail elasticity provides a tractable way to analyse the behaviour of relative tails. Our primitive condition is log-log concavity of the density. Verbally, log-log concavity means that the elasticity of the density with respect to type is decreasing. Equivalently, the log density is concave when type is measured on a log scale. The term log-log concavity is meant to emphasise the multiplicative nature of the problem. Selection is often expressed in terms of ratios, such as productivity relative to a threshold, rather than additive distances from a threshold. The same multiplicative structure appears in many economic objects generated by selection, such as markups defined as ratios of prices to marginal costs, returns defined as ratios of payoffs to initial investments, and aggregate wedges built from proportional changes rather than level differences. This structure makes log-log concavity the natural analogue of ordinary log-concavity. The connection with log-concavity is useful because log-concavity has been widely used to study truncated distributions in additive settings (Bagnoli and Bergstrom, 2005). Our focus is instead on multiplicative truncation. We study how distributional curvature shapes relative tail distributions and the aggregate objects that depend on them.

The paper develops a set of theoretical results. First, building on the notion of an increasing generalized failure rate (IGFR), introduced by Lariviere and Porteus (2001) and developed further by Lariviere (2006), we show that IGFR is equivalent to a monotone ordering of relative tail distributions. Higher thresholds shift the distribution of relative types downward in the first-order stochastic order. Second, we show that log-log concavity, or decreasing density elasticity, provides a primitive and easily verifiable sufficient condition for IGFR. Finally, our main result, Theorem 1, establishes that decreasing density elasticity is necessary and sufficient for a general monotonicity property of ratios of weighted upper-tail integrals. These ratios arise naturally when an aggregate distortion or wedge is measured by comparing two truncated aggregates, such as revenues relative to costs, concentration relative to sales, or weighted tail losses relative to tail mass. The theorem shows that if the relative weight $j_1(u)/j_2(u)$ is increasing in relative type u, then the corresponding aggregate ratio decreases with the selection threshold. This result is the formal link between a primitive distributional condition and monotone aggregate comparative statics.

We also emphasise that, in this context, the Pareto distribution is a special benchmark. It generates a relative tail distribution that is invariant to the threshold level. As a result, composition effects vanish and aggregate elasticities become constant. This property greatly simplifies aggregation in models of international trade.[1] While analytically convenient, Pareto tails restrict the range of equilibrium responses and can obscure economically relevant mechanisms. Our analysis provides primitive conditions under which threshold-dependent aggregation remains tractable beyond the Pareto benchmark.

We use several applications to illustrate the economic content of the theory. These applications show how the same distributional logic organises comparative statics in heterogeneous-firm trade, financial accelerator models, job search, risk management, variable markups, market concentration, and endogenous default. Across these settings, tail curvature determines how selection margins respond to changes in thresholds. Through this channel, it affects aggregate productivity, markups, default risk, average wages, trade elasticities, market concentration, tail risk, and other aggregate outcomes.

In heterogeneous-firm models, threshold-based selection arises naturally in decisions such as entry into export markets. In the canonical model of Melitz (2003), only firms with sufficiently high productivity can cover the fixed costs of exporting and therefore select into foreign markets. In Helpman et al. (2004), firms choose between exporting and foreign direct investment, again based on productivity and fixed costs. In Melitz and Ottaviano (2008), selection into exporting is shaped by market size and the intensity of competition. Related macroeconomic models with endogenous entry and product variety, such as Edmond et al. (2023), provide aggregation results that summarize the welfare implications of micro-level markup heterogeneity. In this class of models, our results clarify when selection changes aggregate productivity and trade elasticities, and when Pareto invariance suppresses these composition effects.

In macro-financial models, selection arises through endogenous default. In Carlstrom and Fuerst

[1] Arkolakis et al. (2012) show that, under Pareto productivity, the gains from trade in Melitz-type models can be summarized by the same sufficient statistics as in standard trade models. Head et al. (2014) show that this equivalence breaks down under non-Pareto distributions, as truncated moments depend on the selection threshold and the constant-elasticity structure no longer holds.

(1997), the productivity threshold determines whether a firm defaults and, through bankruptcy costs, contributes to shock propagation. In Bernanke et al. (1999), the distribution of firm outcomes around the default threshold amplifies aggregate shocks through the financial accelerator mechanism. Subsequent work emphasises the role of changes in risk. Christiano et al. (2014) show that an increase in the dispersion of idiosyncratic shocks raises default risk, while Arellano et al. (2019) argue that higher firm-level volatility accounts for a large part of the decline in employment during the 2007–09 recession. Cooke and Damjanovic (2020) apply this mechanism to endogenous firm entry, and Damjanovic et al. (2020) study how default risk and policy interventions interact with the structure of financial intermediation. Our framework isolates the distributional component of these mechanisms by showing how the shape of the tail affects default thresholds and aggregate wedges.

The remainder of the paper is organized as follows. Section 2 develops the analysis of increasing generalized failure rates and illustrates their role in several standard economic frameworks with endogenous selection. Section 3 introduces log-log concavity of the density, presents the main theorem on ratios of upper-tail integrals, and applies the result to aggregate markups, market concentration, and default risk. Section 4 concludes.

2 Generalized Failure Rate

This section develops the analytical role of the generalized failure rate in models with endogenous selection. We consider heterogeneous-agent environments in which equilibrium outcomes are characterized by threshold rules, so that aggregate responses depend on the distribution of characteristics relative to the threshold. The generalized failure rate, introduced by Lariviere and Porteus (2001) and developed further by Lariviere (2006), provides a convenient way to characterize how the relative tail distribution changes as the threshold moves. We extend this line of analysis by showing that an increasing generalized failure rate is equivalent to first-order stochastic ordering of the relative tail distribution.

2.1 Definition and properties

Let z denote a heterogeneous characteristic with density $f(z)$ and cumulative distribution function $F(z)$ defined on $[z_{\min}, b)$, where $z_{\min} \geq 0$ and $b \leq \infty$. The ordinary failure rate is $h(z) = \frac{f(z)}{1-F(z)}$. Following Lariviere and Porteus (2001), the generalized failure rate is defined as $h_g(z) = zh(z) = \frac{zf(z)}{1-F(z)}$. A distribution has an increasing generalized failure rate, or is IGFR, if $h_g(z)$ is weakly increasing on the relevant support.

In our setting, $h_g(z)$ has a direct selection interpretation. Equilibrium outcomes are characterized by a threshold z^*, so that only agents with $z \geq z^*$ are selected into a particular economic activity. To study the distribution of selected agents' characteristics relative to the threshold, define

$$u = \frac{z}{z^*}.$$

The conditional density of u given $z \geq z^*$ is $f^u_{z^*}(u) = \frac{z^* f(z^* u)}{1-F(z^*)}$. Evaluating this density at the lower boundary $u = 1$ gives $h_g(z^*)$. Thus the generalized failure rate is the boundary density of the relative tail distribution. Equivalently, it is the absolute elasticity of the upper-tail probability with respect to the threshold. This is why IGFR is the natural condition for studying how relative tail distributions change as the selection threshold moves.

In what follows, we analyze the cumulative distribution function of the relative truncated variable $u = z/z^*$ conditional on $z \geq z^*$. The relative tail distribution is

$$F_{z^*}(u) = \Pr\left(\frac{z}{z^*} \leq u \,\middle|\, z \geq z^*\right) = \frac{F(uz^*) - F(z^*)}{1 - F(z^*)}, \qquad \text{for } u \geq 1. \tag{1}$$

This representation describes how the distribution of relative heterogeneity above the selection threshold depends on the cutoff level z^*. To characterize this dependence, we study how $F_{z^*}(u)$ varies with z^*. The following result shows that this comparative static is governed by the generalized failure rate.

Proposition 1. *The generalized failure rate $h_g(z) = \frac{zf(z)}{1-F(z)}$ is increasing in z if and only if the relative tail distribution $F_{z^*}(u)$ is increasing in z^* for every $u \geq 1$.*

Proof. By direct differentiation,

$$\frac{d}{dz^*} F_{z^*}(u) = \frac{1 - F(uz^*)}{z^*\big(1 - F(z^*)\big)} \left[\frac{uz^* f(uz^*)}{1 - F(uz^*)} - \frac{z^* f(z^*)}{1 - F(z^*)}\right].$$

Hence, $F_{z^*}(u)$ is increasing in z^* for all $u \geq 1$ if and only if

$$\frac{uz^* f(uz^*)}{1 - F(uz^*)} \geq \frac{z^* f(z^*)}{1 - F(z^*)},$$

which is equivalent to thegeneralized failure rate being increasing in z. This completes the proof. □

Proposition 1 has an important implication. If the generalized failure rate is increasing, then, as the threshold z^* rises, the conditional CDF F_{z^*} increases pointwise. Equivalently, the relative tail becomes smaller in the sense of first-order stochastic dominance. In particular, for any increasing function $\varphi(u)$, the expectation of φ under F_{z^*} decreases with z^*. We denote such expectations by $E_{z^*}[\varphi(u)]$.

The next figure provides a simple illustration of this ordering. It plots the relative tail distribution for a lognormal density at three different threshold values. The upward shift of the conditional CDF as the threshold rises is the graphical counterpart of the stochastic dominance result in Proposition 1.

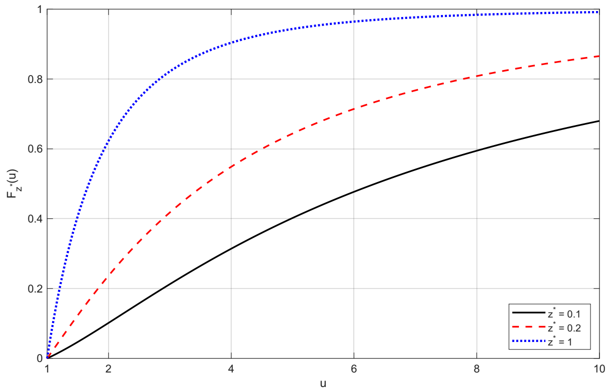

Figure 1: Relative tail distribution $F_{z^*}(u)$ for a lognormal distribution with $\mu = -1/2$ and $\sigma = 1$.

Figure 1 shows that higher thresholds place more probability mass near the lower bound, $u = 1$. This relative-tail compression effect is the central distributional mechanism in the paper.

2.1.1 Pareto distribution is a special case

The role of the generalized failure rate is particularly transparent under a Pareto distribution. Suppose z follows an unbounded Pareto distribution on $[z_{\min}, \infty)$ with shape parameter $\kappa > 0$, so that

$$1 - F(z) = \left(\frac{z_{\min}}{z}\right)^{\kappa}. \tag{2}$$

In this case, the generalized failure rate is constant, $h_g(z) = \frac{zf(z)}{1-F(z)} = \kappa$, and the relative tail distribution is therefore independent of the selection threshold. Indeed, for $u = z/z^*$,

$$F_{z^*}(u) = \frac{F(uz^*) - F(z^*)}{1 - F(z^*)} = 1 - u^{-\kappa},$$

which does not depend on z^*.

As a result, all relative truncated moments, whenever they exist, are invariant to the threshold. This invariance explains why Pareto assumptions greatly simplify heterogeneous-agent models: selection changes the mass of active agents but not their relative composition. By contrast, when the generalized failure rate is not constant, selection generally changes the relative tail distribution, generating composition effects and richer aggregate comparative statics.

2.2 Economic Applications

This section applies the theoretical results developed above to a range of standard economic environments with endogenous selection. In these settings, aggregate outcomes depend on the distribution of characteristics among agents who remain active after a threshold rule is applied. The normalized tail representation and the behavior of the generalized failure rate provide convenient tools for deriving comparative statics of key aggregate objects. The subsections below illustrate applications of Proposition 1 in environments where the increasing generalized failure rate property governs the behavior of the relative tail distribution.

2.2.1 Average outcomes and relative-tail compression

Consider an economy in which firms differ in productivity, and only firms with productivity above an equilibrium threshold, z^*, remain active. Average productivity among active firms is

$$E(z \mid z > z^*) = \frac{\int_{z^*}^{\infty} z f(z)\, dz}{1 - F(z^*)}.$$

Using the substitution $u = z/z^*$, this expression becomes

$$E(z \mid z > z^*) = z^* \frac{\int_1^{\infty} u z^* f(u z^*)\, du}{1 - F(z^*)} = z^* \int_1^{\infty} u\, dF_{z^*}(u) = z^* E_{z^*}[u],$$

where $F_{z^*}(u)$ is the relative tail distribution defined in (1) and $E_{z^*}[\cdot]$ denotes expectation with respect to this distribution.

This decomposition separates two effects of a higher threshold. The first is a direct scale effect: active firms are selected from a higher region of the productivity distribution. The second is a composition effect: the distribution of productivity relative to the threshold may itself change. By Proposition 1, if the generalized failure rate is increasing, then $F_{z^*}(u)$ increases pointwise with z^*. Hence relative productivity among active firms shifts downward in the first-order stochastic order, and $E_{z^*}[u]$ must decrease with z^*. Tighter selection therefore raises the absolute productivity threshold but compresses productivity relative to that threshold.

The same composition mechanism arises in labor-market search models. In the McCall job search framework, workers accept job offers only if the wage z exceeds a reservation wage z^*. The average accepted wage is therefore the conditional mean above the threshold, $\overline{z}(z^*) = z^* E_{z^*}[u]$. Under an increasing generalized failure rate, the expected relative wage $E_{z^*}[u]$ declines as the reservation wage rises. Thus the average accepted wage increases more slowly than the reservation wage itself, reflecting compression of the relative upper tail of the wage-offer distribution.

For Pareto productivity with $\kappa > 1$, the composition effect is absent and $E_{z^*}[u] = \frac{\kappa}{\kappa-1}$. Hence, average productivity is simply proportional to z^*. Outside the Pareto case, the generalized failure rate determines how relative composition changes as selection becomes more stringent.

2.2.2 Selection and gains from trade in the Melitz model

A central application of endogenous selection with productivity cutoffs is the heterogeneous-firm trade model of Melitz (2003). The model uses the Dixit and Stiglitz (1977) monopolistic competition framework, which implies downward-sloping demand curves with elasticity of substitution $\sigma > 1$. If a firm with productivity z serves a given market, its revenue, $r(z)$, and operating profit, $\pi(z)$, are proportional to $z^{\sigma-1}$. Writing Π for the market-specific demand shifter, which depends on market size, source-country characteristics, the elasticity of substitution, and non-trade costs, and writing τ for variable trade costs, we can summarize export revenue and profit as

$$r_x(z) = \sigma\Pi\left(\frac{z}{\tau}\right)^{\sigma-1}, \qquad \pi_x(z) = \frac{r(z)}{\sigma} - f_x,$$

where f_x is the fixed cost of serving the foreign market.[2]

The productivity cutoff z^* for exporting is endogenously determined by the zero-profit condition,

$$\pi(z^*) = 0 \qquad \Longleftrightarrow \qquad z^* = \tau\left(\frac{f_x}{\Pi}\right)^{\frac{1}{\sigma-1}}. \tag{3}$$

Hence only firms with $z \geq z^*$ export. Aggregate export revenue is therefore

$$R = \int_{z^*}^{\infty} r(z)f(z)\,dz = \sigma\Pi\left(\frac{z^*}{\tau}\right)^{\sigma-1}\int_{z^*}^{\infty}\left(\frac{z}{z^*}\right)^{\sigma-1} f(z)\,dz.$$

Using the cutoff condition (3), this can be rewritten as

$$R = \sigma f_x\big[1 - F(z^*)\big]E_{z^*}\left(u^{\sigma-1}\right),$$

where $u = z/z^*$, and $E_{z^*}\left(u^{\sigma-1}\right) = \frac{\int_{z^*}^{\infty}(z/z^*)^{\sigma-1}f(z)\,dz}{1-F(z^*)}$.

This decomposition separates three distinct forces. A rise in the fixed export cost f_x increases the cutoff z^*, reduces the mass of exporters $1-F(z^*)$, and increases the average relative productivity

[2] Domestic revenue and profits (equations (4) and (5) in Melitz (2003)) are given by,

$$r(z) = R\,(P\rho z)^{\sigma-1} \quad \text{and} \quad \pi(z) = \frac{r(z)}{\sigma} - f$$

where P is the price index, $\sigma/(\sigma-1) = 1/\rho$ is the fixed markup, $R = \int_{z^*}^{\infty} r(z)f(z)dz$, and z is idiosyncratic productivity. Export revenue is likewise, $r_x(z) = R_x\left(P_x\rho\frac{z}{\tau}\right)^{\sigma-1}$, where τ are variable trade costs. By symmetry across countries, we can write $P_x = P$ and $R_x = R = L$, which is country size. In this case, define $\Pi \equiv (1-\rho)(\rho P)^{\rho/(1-\rho)}L = \Pi_x$, which implies, $r_x(z) = \sigma\Pi\left(\frac{z}{\tau}\right)^{\sigma-1}$. $\pi_x(z)$ follows from this.

of surviving exporters through $E_{z^*}(u^{\sigma-1})$.

Following Chaney (2008), we can analyse the elasticity of export sales with respect to variable trade costs τ and fixed trade costs f_x. If productivity is Pareto, then the truncated moment $E_{z^*}(u^{\sigma-1})$ does not depend on the cutoff z^*. In addition, the survival probability has the power form $1 - F(z^*) = (z^*)^{-\kappa}$. Since $z^* = \tau \left(\frac{f_x}{\Pi}\right)^{\frac{1}{\sigma-1}}$, aggregate export revenue satisfies $R \propto f_x^{1-\kappa/(\sigma-1)} \tau^{-\kappa}$. Therefore,

$$-\frac{d \log R}{d \log f_x} = \frac{\kappa}{\sigma - 1} - 1 \quad \text{and} \quad -\frac{d \log R}{d \log \tau} = \kappa. \tag{4}$$

Formula (4) replicates the result in Proposition 2 of Chaney (2008). The elasticities are constant, and the trade elasticity with respect to variable trade costs is independent of the elasticity of substitution between varieties, σ.

This conclusion depends critically on the Pareto assumption. If productivity is not Pareto, then the truncated moment $E_{z^*}u^{\sigma-1}$ generally varies with the cutoff. In that case,

$$\frac{d \log R}{d \log f_x} = 1 + \left(\frac{d \log \left[1 - F(z^*)\right]}{d \log z^*} + \frac{d \log E_{z^*}u^{\sigma-1}}{d \log z^*} \right) \frac{d \log z^*}{d \log f_x}, \tag{5}$$

and, since $\frac{d \log z^*}{d \log f_x} = \frac{1}{\sigma-1}$ and $\frac{d \log z^*}{d \log \tau} = 1$, this gives

$$-\frac{d \log R}{d \log f_x} = -1 + \left(h_g\left(z^*\right) - \frac{d \log E_{z^*}u^{\sigma-1}}{d \log z^*} \right) \frac{1}{\sigma - 1}, \tag{6}$$

$$-\frac{d \log R}{d \log \tau} = -\frac{d \log R}{d \log z^*} \frac{d \log z^*}{d \log \tau} = h_g\left(z^*\right) - \frac{d \log E_{z^*}u^{\sigma-1}}{d \log z^*}. \tag{7}$$

Thus, outside the Pareto case, trade elasticities are no longer constant, as they depend on how the relative tail distribution changes with the cutoff. If $h_g\left(z^*\right)$ is increasing, we can apply Proposition 1, which implies $\frac{d \log E_{z^*}u^{\sigma-1}}{d \log z^*} < 0$. Thus, under the monotonicity conditions developed above, the elasticity of export revenue with respect to both fixed and variable trade costs increases with the cutoff. Moreover, the trade elasticity is no longer independent of the elasticity of substitution σ (a key implication from Chaney's (2008) analysis). In Section 3 we formulate sufficient conditions under which the elasticity changes systematically with σ.

Our results have two practical applications both of which rely on non-Pareto distributions. Helpman, Melitz, and Rubinstein (2008) use CES utility together with a bounded Pareto distribution to obtain trade elasticities that vary systematically across country pairs. The upper-truncated Pareto distribution has an increasing generalized failure rate. Let b denote the upper bound of the truncated Pareto distribution. Then $h_g(z) = \kappa/[1 - (z/b)^\kappa]$, which is increasing in

z. Therefore, the bounded Pareto distribution satisfies the assumptions of Proposition 1. In this case, trade elasticities are no longer constant, even though the distribution remains Pareto over its bounded support. Head, Mayer, and Thoenig (2014) assume productivity is log-normal. They are interested in the result of Arkolakis, Costinot, and Rodríguez-Clare (2012), who show that in a broad class of quantitative trade models, aggregate welfare gains can be expressed using only two sufficient statistics: the domestic expenditure share and the trade elasticity.[3] Head, Mayer, and Thoenig (2014) show that this result breaks down and welfare depends not only on the domestic trade share, but also on endogenous entry and selection, in particular the mass of entrants and the behaviour of the relevant tail of the productivity distribution.

2.2.3 The generalized failure rate and BGG financial accelerator

The generalized failure rate also plays a central role in the financial accelerator model of Bernanke, Gertler, and Gilchrist (1999). In this framework, a risk-neutral lender finances an entrepreneur whose project return is subject to an idiosyncratic productivity shock z drawn from a distribution F with density f. Because entrepreneurs have limited liability and monitoring is costly, default occurs when realized output is insufficient to meet the promised repayment. Let z^* denote the default threshold, below which the entrepreneur defaults and above which the lender receives the promised repayment.

The lender's expected gross return per unit of lending can be written as a function of the default threshold[4] as

$$\Psi(z^*) = (1-\mu)\int_0^{z^*} z f(z)\,dz + z^*\big[1 - F(z^*)\big].$$

Here $\mu \in (0,1)$ is the exogenous monitoring-cost parameter. Optimal contract choice involves selecting the threshold z^* that maximizes $\Psi(z^*)$. The first-order condition is

$$\Psi'(z^*) = 0 \quad \Longleftrightarrow \quad \frac{z^* f(z^*)}{1 - F(z^*)} = \frac{1}{\mu}, \tag{8}$$

Thus the equilibrium default threshold is pinned down by the generalized failure rate of the productivity distribution. Contract determination therefore depends on the behaviour of the upper tail through a simple sufficient statistic.

Bernanke, Gertler, and Gilchrist (1999) assume that the generalized failure rate is increasing (equation 3.1, p. 1349). However, monotonicity alone does not guarantee the existence of a

[3] This result implies that different microeconomic models may generate the same aggregate gains from trade when they share the same values of these two statistics. In this sense, the Melitz model does not necessarily change the quantitative evaluation of aggregate welfare gains relative to simpler trade models.

[4] See Bernanke, Gertler, and Gilchrist (1999, p. 1381), using $\Psi := \Gamma - \mu G$.

finite solution. To ensure existence for any $\mu \in (0,1)$ under the usual continuity assumptions, it is further required that

$$\lim_{z\to\infty} h_g(z) = \infty.$$

Not every distribution with an increasing generalized failure rate satisfies this condition. A simple counterexample is the Lomax distribution, $F(z) = 1 - \left(1 + \frac{z}{\lambda}\right)^{-\kappa}$ with $z \geq 0$. Then

$$h_g(z) = \frac{\kappa z}{\lambda + z},$$

which is strictly increasing but bounded, with $\lim_{z\to\infty} h_g(z) = \kappa < \infty$. Consequently, if $\kappa < \frac{1}{\mu}$, the optimality condition admits no finite solution. This example illustrates that the upper-tail behaviour of the generalized failure rate is a key determinant of equilibrium contract existence and uniqueness in macro-finance models with endogenous default.

Finally, following BGG, we can also explain why the unbounded Pareto distribution is not suitable in this setting. For an unbounded Pareto distribution, $h_g(z) = \kappa$, which is independent of z. In this case, equation (8) either has no solution if $\kappa \neq \frac{1}{\mu}$ or the function $\Psi(z^*)$ is constant if $\kappa = \frac{1}{\mu}$. TThus, the Pareto case does not deliver a well-defined interior contract threshold in the BGG model, which requires the generalized failure rate to be both increasing and unbounded.

2.2.4 Expected Shortfall, Value at Risk, and relative tails.

A closely related application arises in the measurement of tail risk in financial stability. Expected Shortfall (ES), also known as Conditional Value at Risk, measures the expected size of losses conditional on losses exceeding a high quantile. This distinction is important in systemic-risk analysis because financial stability depends not only on whether losses enter the tail, but also on the severity of losses conditional on being in the tail. For example, Acharya et al. (2017) use Expected Shortfall as a building block for measuring the exposure of financial institutions to severe aggregate downturns.

The regulatory relevance of ES is also reflected in the post-crisis Basel market-risk framework. Under the revised Basel III market-risk standards, the Internal Models Approach relies on Expected Shortfall models rather than VaR as the central tail-risk metric for modellable risk factors. The resulting internal-model capital requirement enters the calculation of market-risk risk-weighted assets. Thus, ES is not only a theoretical object in tail-risk analysis; it is also embedded in the regulatory measurement of bank capital requirements.

Let losses z follow a continuous distribution F with density f. For a tail probability α, define Value at Risk as the threshold $VaR_\alpha = z^*$ satisfying $1 - F(z^*) = \alpha$. The Expected Shortfall associated with this threshold is

$$ES_\alpha = \mathbb{E}[z \mid z \geq z^*] = \frac{\int_{z^*}^{\infty} z f(z)\, dz}{1 - F(z^*)}.$$

Introducing the relative tail variable $u = \frac{z}{z^*}$, we can rewrite Expected Shortfall as $ES_\alpha = z^* \mathbb{E}_{z^*}[u]$. Since $z^* = VaR_\alpha$, this gives

$$\frac{ES_\alpha}{VaR_\alpha} = \mathbb{E}_{z^*}[u].$$

This expression shows that the ratio of Expected Shortfall to Value at Risk is exactly a relative-tail moment. Hence, our framework directly characterizes how this ratio changes as the confidence level changes. If the generalized failure rate is increasing, then the relative tail distribution shifts downward in the first-order stochastic order as the threshold z^* rises. Consequently, $\mathbb{E}_{z^*}[u]$ declines with z^*, implying that Expected Shortfall grows less than proportionally with Value at Risk.

Acharya et al. (2017) obtain a particularly simple version of this relationship under Pareto-tail assumptions[5]. In their setting, tail shocks follow normalized Pareto distributions with tail exponent κ. Under this assumption, $\mathbb{E}_{z^*}[u] = \frac{\kappa}{\kappa - 1}$, and hence

$$ES_\alpha = \frac{\kappa}{\kappa - 1} VaR_\alpha.$$

Thus, the Pareto case delivers a constant ES/VaR ratio independent of the confidence level. Our contribution is to show how this familiar Pareto proportionality extends to more general distributions. Outside the Pareto benchmark, the ratio ES_α / VaR_α is no longer constant. Instead, its monotonicity is governed by the relative tail distribution, or equivalently by the generalized failure rate.

This observation is relevant for market-risk measurement. Under the Basel market-risk framework, the Internal Models Approach relies on Expected Shortfall as the central tail-risk measure. Banks using internal models are required to calculate Expected Shortfall at a one-tailed 97.5 percent confidence level. However, the framework does not prescribe a particular parametric distribution for IMA Expected Shortfall. Instead, banks may use their own internal models, provided that these models are approved by supervisors and pass regulatory tests designed to assess whether they capture trading losses reliably. Thus, the behaviour of the tail beyond the VaR threshold remains a modelling issue. This makes the distributional properties studied here directly relevant for the design and evaluation of IMA models.

[5] See the first paragraph on page 14.

2.2.5 Competition and Financial Risk

Proposition 1 provides a useful way to analyse the trade-off between competition and financial stability in models with endogenous default. Arellano et al. (2019) consider an economy in which firms borrow working capital before the realization of idiosyncratic demand, distributed according to F. Firms are monopolistically competitive and face CES demand with elasticity $\frac{1}{1-\eta}$, where $0 < \eta < 1$. The equilibrium markup is $\frac{1}{\eta}$, so a higher value of η corresponds to a more competitive environment. Firms hire labour and choose their production scale before the idiosyncratic shock is realized. A larger production scale raises the planned probability of default, while a lower markup reduces the profit buffer that protects firms against adverse shocks. Thus, greater competition can induce firms to operate at a larger scale and take on more default risk.

Under limited liability, profit maximization implies that expected revenue is set as a markup over expected marginal cost, while the scale of investment determines the probability of default. In a symmetric equilibrium, the default threshold z^* satisfies

$$\int_{z^*}^{\infty} \left(\frac{z}{z^*}\right)^{\eta} f(z)\, dz = \frac{1}{\eta}\big(1 - F(z^*)\big). \tag{9}$$

Equivalently, define

$$H(z^*, \eta) = \frac{\int_{z^*}^{\infty} \left(\frac{z}{z^*}\right)^{\eta} f(z)\, dz}{1 - F(z^*)} - \frac{1}{\eta} = \mathbb{E}_{z^*}[u^{\eta}] - \frac{1}{\eta}. \tag{10}$$

Then the equilibrium default threshold is characterized by $H(z^*, \eta) = 0$.

The unbounded Pareto case illustrates that a solution may fail to exist or may fail to be unique. If F is Pareto with curvature κ, then $H(z^*, \eta) = \frac{\kappa}{\kappa-\eta} - \frac{1}{\eta}$ which is independent of z^*. Thus, the equilibrium condition cannot generally pin down a finite default threshold. If $H(z^*, \eta) > 0$ for all z^*, the borrower has an incentive to take arbitrarily large risk. If instead $H(z^*, \eta) < 0$, then there is no interior default threshold. The latter case is less relevant for the risk-taking mechanism emphasized here.

We can state the existence, uniqueness, and comparative-statics result more generally.

Proposition 2. *Consider a distribution F on $[z_{\min}, b)$ with an increasing generalized failure rate. Suppose also that $H(z_{\min}, \eta) > 0$ and*

$$\lim_{z^* \to b} h_g(z^*) = +\infty.$$

Then there exists a unique equilibrium default threshold z^* satisfying $H(z^*, \eta) = 0$. Moreover, the default threshold is increasing in η. Hence, greater competition raises the probability of default.

Proof. First, by L'Hôpital's rulewe obtain $\lim_{z^* \to b} H(z^*, \eta) = 1 - 1/\eta < 0$ [6]. Since $H(z_{\min}, \eta) > 0$, continuity implies that at least one solution to $H(z^*, \eta) = 0$ exists. Uniqueness follows because $H(z^*, \eta)$ is strictly decreasing in z^* by Proposition 1. Hence $\partial H(z^*, \eta)/\partial z^* < 0$. Next,

$$\frac{\partial H(z^*, \eta)}{\partial \eta} = \frac{\int_{z^*}^{\infty} \left(\frac{z}{z^*}\right)^{\eta} \log\left(\frac{z}{z^*}\right) f(z)\, dz}{1 - F(z^*)} + \frac{1}{\eta^2} > 0. \tag{11}$$

Therefore, by the implicit function theorem,

$$\frac{dz^*}{d\eta} = -\frac{\partial H/\partial \eta}{\partial H/\partial z^*} > 0.$$

Thus, an increase in competition, represented by a higher value of η, raises the equilibrium default threshold. Since default occurs for realizations below z^*, the probability of default, $F(z^*)$, also increases.

This positive relationship between competition and default captures a central trade-off in financial-sector models. On the one hand, stronger competition lowers markups, makes credit cheaper, and can increase investment. On the other hand, thinner margins make banks and firms more vulnerable to shocks, especially systemic non-diversifiable shocks, because profits act as a buffer against losses and provide partial hedging. Damjanovic, Damjanovic, and Nolan (2020) emphasize this mechanism in an investment-banking setting. They show that greater competition increases risk-taking, while tighter macroprudential policy can offset this effect by increasing markups and reducing default probabilities. Thus, macroprudential policies that raise capital requirements or restore profit margins may be desirable when excessive competition makes margins too thin.

Competition-induced increases in default rates, together with the associated resolution costs emphasized by Carlstrom and Fuerst (1997, 1998), may generate significant welfare losses when idiosyncratic shocks become more dispersed. Christiano, Motto, and Rostagno (2014) refer to such an increase in firm-level uncertainty as a risk shock. In the present framework, this shock matters because firms choose employment and production before the realization of idiosyncratic

[6] Indeed,

$$\lim_{z^* \to b} \frac{\int_{z^*}^{b} z^{\eta} f(z)\, dz}{(z^*)^{\eta}\big(1 - F(z^*)\big)} = \lim_{z^* \to b} \frac{-(z^*)^{\eta} f(z^*)}{\eta (z^*)^{\eta - 1}\big(1 - F(z^*)\big) - (z^*)^{\eta} f(z^*)} = \lim_{z^* \to b} 1 + \frac{1}{\eta \frac{z^* f(z^*)}{1 - F(z^*)} - 1}$$

demand. A larger production scale increases the planned probability of default. Hence, when the distribution of shocks becomes more dispersed, firms may choose allocations that involve higher default risk. The resulting increase in expected resolution costs is not fully internalized by individual firms.

In economies with exogenous entry, this externality may justify a positive adjustment of profit taxation in response to a risk shock. Higher profit taxation lowers expected post-tax profits and reduces entry. This weakens competition, raises equilibrium markups, and increases the profit buffer that protects firms against adverse shocks. As a result, firms choose a lower planned probability of default, and expected resolution costs decline. Cooke and Damjanovic (2020) use this mechanism to show that an increase in profit taxation can be welfare-improving when a risk shock hits the economy.

Posted prices and virtual valuations Finally, a closely related object appears in auction theory; see Myerson (1981). Let x denote the seller's value of the good, and let z denote the bidder's valuation. The seller does not observe the realization of z, but knows its distribution F, with density f. Consider a take-it-or-leave-it offer at price z^*. The bidder accepts whenever her valuation is at least z^*, so the seller's expected payoff is

$$\Pi(z^*) = (z^* - x)\big(1 - F(z^*)\big).$$

An interior optimum satisfies

$$x = \phi(z^*) = z^* \left(1 - \frac{1 - F(z^*)}{z^* f(z^*)}\right), \tag{12}$$

where $\phi(z)$ is Myerson's virtual valuation function. Thus the seller chooses the price at which the bidder's virtual valuation equals the seller's own value of the good. This condition can be rewritten as

$$\frac{z^*}{x} = 1 + \frac{1}{h_g(z^*) - 1}.$$

This expression gives a simple interpretation of the generalized failure rate in the posted-price problem. At the optimum, the $h_g(z^*)$ determines the wedge between the seller's value x and the price z^* offered to the bidder. If $h_g(z^*)$ is large, the upper tail above the posted price is thin in relative terms, so raising the price sharply reduces the probability of sale. The optimal price is then close to the seller's value. By contrast, when $h_g(z^*)$ is close to one, the upper tail is thick, and the seller can optimally set a price far above her own value.

This representation also gives a simple comparative-static implication. Under IGFR, higher optimal thresholds are associated with larger values of $h_g(z^*)$. Therefore, higher seller values are associated with lower proportional markups whenever they lead to higher optimal posted prices.

The same representation also links the generalized failure rate to regularity. For a positive seller value, $x > 0$, condition 12 implies $h_g(z^*) > 1$. Hence the economically relevant posted-price threshold lies in the region where IGFR supports monotonicity of the virtual valuation.

This example illustrates that the same tail elasticity that governs relative-tail selection also appears in threshold-pricing problems. The main focus of the paper, however, is on aggregate objects generated by endogenous selection.

The generalized failure rate therefore provides a sufficient statistic for characterizing selection and threshold determination in the applications considered in this section. However, by itself it does not deliver monotonicity results for broader classes of aggregate objects. In the next section, we introduce log-log concavity, a primitive distributional condition that yields general monotonicity results for aggregate values in models with endogenous selection.

3 Log-Log Concavity and aggregate ratios

The analysis so far has shown that equilibrium responses to threshold changes are governed by the behaviour of relative tail distributions. The generalized failure rate provides a compact way to characterize these responses, but it is defined in terms of the survival function and may therefore be less convenient as a primitive condition. We now introduce log-log concavity, a condition imposed directly on the density. This condition is easy to verify in applications and provides a structural foundation for monotone selection effects.

Log-log concavity is useful because many aggregate objects in selection models can be written as ratios of upper-tail integrals. These objects include average productivity, trade elasticities, market concentration, aggregate markups, and default-related wedges. The results below show that a simple restriction on the elasticity of the density generates systematic comparative statics for this broad class of aggregate outcomes.

3.1 Definition and theoretical properties

We begin with a formal definition.

Definition 3. *Let f be a positive, continuously differentiable probability density function supported on $[z_{\min}, b)$, where $z_{\min} \geq 0$ and $b \leq +\infty$. The density $f(z)$ is said to be log-log concave* *if the function $x \mapsto \log f(e^x)$ is concave on its domain. Equivalently, f is log-log concave if the elasticity of the density,*

$$\varepsilon_f(z) = \frac{d \log f(z)}{d \log z} = \frac{z f'(z)}{f(z)}, \tag{13}$$

is decreasing in z.

This condition is known in stochastic-order theory through its equivalence to a likelihood-ratio ordering of scaled random variables. In particular, Shaked and Shanthikumar (2007, Theorem 1.C.29) show that $aX \leq_{lr} X$ for all $0 < a < 1$ if and only if $x \mapsto \log f(e^x)$ is concave. We refer to this condition as log-log concavity because it requires concavity of the log-density with respect to log type.

Log-log concavity provides a primitive structural condition for an increasing generalized failure rate. To see the connection with existing results, let Z be a positive random variable with density f, and define $X = \log Z$. The density of X is $g(x) = e^x f(e^x)$. Hence, $\log g(x) = x + \log f(e^x)$. Thus log-log concavity of f is exactly log-concavity of g, the density of the log-transformed variable.

The next proposition therefore follows from two known results. Bagnoli and Bergstrom (2005) show that a log-concave density has an increasing failure rate, while Lariviere (2006) shows that a positive random variable has an increasing generalized failure rate if and only if its logarithm has an increasing failure rate. We state the result explicitly because it translates these observations into the density-elasticity language used throughout the paper.

Proposition 4. *Suppose that the density f is log-log concave. Then the generalized failure rate is increasing*

Proof. Let Z have density f, and set $X = \log Z$. The preceding argument shows that log-log concavity of f is equivalent to log-concavity of the density of X. Therefore X has an increasing failure rate by Bagnoli and Bergstrom (2005). By Lariviere (2006), this is equivalent to Z having an increasing generalized failure rate. Hence $h_g(z)$ is increasing in z. □

Constant density elasticity implies a constant generalized failure rate and therefore Pareto tails. However, an increasing generalized failure rate does not in general imply monotone density elasticity, and simple counterexamples can be constructed.

Example 5 (Increasing generalized failure rate without log-log concavity)**.** *The converse of Proposition 4 is false. Let $X \geq 0$ have survival function*

$$1 - F(x) = \exp\left(-x - \frac{x^3}{3}\right).$$

Its failure rate is $h(x) = 1 + x^2$*, which is increasing. However, the density* $f(x) = (1 + x^2)\exp\left(-x - \frac{x^3}{3}\right)$ *is not log-concave, since* $\log f(x) = \log(1+x^2) - x - \frac{x^3}{3}$ *has a positive second derivative at* $x = 0$*.*
Now define $Z = e^X$*. Then* Z *is supported on* $[1, \infty)$*, and its generalized failure rate is increasing in* z*. However, the density of* Z *is not log-log concave.*
Finally, Z *has finite moments of all orders, since for every finite* m*,* $E_z[Z^m] = E_x[e^{mx}] < \infty$ *as* $\exp(-x^3/3)$ *dominates* $\exp(mx)$ *as* $x \to \infty$*.*

The property needed for the selection results above is an increasing generalized failure rate, but direct verification can be cumbersome. Log-log concavity provides a convenient sufficient condition because it is equivalent to decreasing density elasticity, which can often be verified by differentiating the density. It therefore provides a simple primitive for monotone selection responses.

Table 1 below reports density functions and corresponding density elasticities for several commonly used distributions.

Table 1: Density functions and density elasticities of commonly used distributions

Distribution	Density $f(z)$	Elasticity $\varepsilon_f(z)$
Pareto (κ)	$\kappa z_m^\kappa z^{-(\kappa+1)}$	$-(\kappa+1)$
Fréchet (α, s)	$\alpha s^\alpha z^{-(\alpha+1)} \exp[-(s/z)^\alpha]$	$-(\alpha+1) + \alpha\left(\frac{s}{z}\right)^\alpha$
Lognormal (μ, σ)	$\frac{1}{z\sigma\sqrt{2\pi}} \exp\left[-\frac{(\log z - \mu)^2}{2\sigma^2}\right]$	$-1 - \frac{\log z - \mu}{\sigma^2}$
Weibull (θ, k)	$k\theta^{-k} z^{k-1} \exp\left[-(z/\theta)^k\right]$	$(k-1) - k(z/\theta)^k$
Gamma (a, b)	$\frac{1}{\Gamma(a)b^a} z^{a-1} e^{-z/b}$	$(a-1) - \frac{z}{b}$
Exponential (λ)	$\lambda e^{-\lambda z}$	$-\lambda z$
Chi-square (ν)	$\frac{1}{2^{\nu/2}\Gamma(\nu/2)} z^{\nu/2-1} e^{-z/2}$	$\left(\frac{\nu}{2} - 1\right) - \frac{z}{2}$
Half-normal (σ)	$\frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left[-\frac{z^2}{2\sigma^2}\right]$	$-\frac{z^2}{\sigma^2}$
Beta (a, b), $b \geq 1$	$\frac{1}{B(a,b)} z^{a-1}(1-z)^{b-1}$	$(a-1) - (b-1)\frac{z}{1-z}$
Lomax/Pareto II (κ, λ)	$\frac{\kappa}{\lambda}\left(1 + \frac{z}{\lambda}\right)^{-(\kappa+1)}$	$-(\kappa+1)\frac{z}{\lambda+z}$
Burr (c, k)	$ckz^{k-1}\left(1 + z^k\right)^{-(c+1)}$	$(k-1) - k(c+1)\frac{z^k}{1+z^k}$
Half-Student-t (ν)	$\frac{2\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)}\left(1 + \frac{z^2}{\nu}\right)^{-(\nu+1)/2}$	$-(\nu+1)\frac{z^2}{\nu+z^2}$

In each of these cases, the elasticity is decreasing in z, implying log-log concavity and hence an increasing generalized failure rate.

The Pareto distribution represents a boundary case in which the elasticity is constant, so that the generalized failure rate is constant and relative tail distributions are invariant to the selection threshold.

The condition of decreasing density elasticity is not automatic. For example, the perturbed Pareto density, $f(z) = Cz^{-\kappa}\exp(\beta/z)$, where C is normalising constant, has $\varepsilon_f(z) = -\kappa - \frac{\beta}{z}$ which is increasing in z. Nevertheless, its tail remains Pareto-like, and its m-th moment is finite if and only if $\kappa > m + 1$. Thus the failure of log-log concavity need not be associated with pathological tail behaviour.

Density elasticity provides a simple way to verify the increasing generalized failure rate condition. It can also be used to study the asymptotic behaviour of the generalized failure rate itself. This is useful in applications with endogenous thresholds, including Bernanke, Gertler, and Gilchrist (1999) and the competition-and-financial-fragility application in Section 2.2.5, where one often needs the generalized failure rate to become sufficiently large in the upper tail. Checking whether the IGFR is unbounded is also useful for verifying the existence of moments. Theorem 2 of Lariviere (2006) implies that an increasing and unbounded generalized failure rate is sufficient for all polynomial moments to be finite. The next proposition shows that this requirement follows from a simple primitive condition, that is the density elasticity tends to $-\infty$.

Proposition 6. *If* $\lim_{z\to\infty} \frac{zf'(z)}{f(z)} = -\infty$, *then* $\lim_{z\to\infty} \frac{zf(z)}{1-F(z)} = +\infty$.

Proof. Consider the reciprocal ratio $\frac{1-F(z)}{zf(z)}$. As $z \to \infty$, both the numerator and the denominator converge to zero. Applying l'Hôpital's rule yields

$$\lim_{z\to\infty} \frac{1 - F(z)}{zf(z)} = \lim_{z\to\infty} \frac{-f(z)}{f(z) + zf'(z)} = \lim_{z\to\infty} \frac{-1}{1 + \frac{zf'(z)}{f(z)}}.$$

Since $\frac{zf'(z)}{f(z)} \to -\infty$, the last expression converges to zero. Therefore, $\lim_{z\to\infty} \frac{zf(z)}{1-F(z)} = +\infty$. □

This is particularly useful in financial-accelerator models, where default thresholds are endogenous and existence may require the generalized failure rate to become arbitrarily large in the upper tail. The proposition provides a simple density-elasticity condition for verifying this requirement.

The next section establishes the main aggregation result of the paper. It shows that log-log concavity translates movements in endogenous thresholds into monotone comparative statics for ratios of aggregate objects formed over the selected upper tail.

3.2 Ratios of integrals and log-log concavity

This section establishes the main theoretical result of the paper. The result concerns ratios of aggregate objects that arise in models with endogenous selection.

In many applications, aggregate outcomes are obtained by integrating individual contributions over the set of agents whose idiosyncratic characteristics lie above an endogenous threshold. After individual types are normalized by the threshold, these aggregate objects can be written as integrals over the relative tail distribution. The weight functions in these integrals represent the economic quantities being aggregated, such as output, profits, losses, repayments, or markups.

Consider a pair of positive weighting functions (j_1, j_2) with $j_i : (1, \infty) \to (0, \infty)$ and define the upper-tail integrals

$$J_i(z^*) = \int_{z^*}^{b} j_i\left(\frac{z}{z^*}\right) f(z)\, dz. \tag{14}$$

Theorem 1 gives the main aggregation result of the paper. It shows that, under log-log concavity, ratios of such upper-tail aggregates move monotonically with the cutoff, if $j_1(u)/j_2(u)$ is increasing in u, then $J_1(z^*)/J_2(z^*)$ is decreasing in z^*.

Thus decreasing density elasticity provides a primitive condition for monotone comparative statics of aggregate ratios formed over the selected upper tail. This covers many economic objects, including average productivity, trade elasticities, repayment rates, loss ratios, market concentration, and aggregate markups.

We impose the following regularity conditions. The density f is strictly positive and continuously differentiable, and the elasticity $\varepsilon_f(z)$ is well defined on $[z_{\min}, b)$. If $b < \infty$, we assume that f has a finite left limit at b. A pair of weighting functions (j_1, j_2) is admissible if the functions are measurable, strictly positive, and the weighted tail integrals are finite.

Theorem 1. *Let f satisfy the regularity conditions above. For $i = 1, 2$, define*

$$J_i(z^*) = \int_{z^*}^{b} j_i\left(\frac{z}{z^*}\right) f(z)\, dz.$$

Then the density elasticity $\varepsilon_f(z) = \frac{zf'(z)}{f(z)}$ is non-increasing if and only if, for every admissible pair of weighting functions (j_1, j_2) such that $j_1(u)/j_2(u)$ is increasing in u, the ratio $J_1(z^*)/J_2(z^*)$ is non-increasing in z^*.

For the unbounded Pareto distribution, the ratio $J_1(z^*)/J_2(z^*)$ is constant and does not depend on z^*.

Proof. We first prove sufficiency. Changing variables to $u = z/z^*$, we obtain

$J_i(z^*) = z^* \int_1^{b/z^*} j_i(u) f(z^* u), du, \qquad i = 1, 2.$

The common factor z^* cancels in the ratio $J_1(z^*)/J_2(z^*)$. Let $r(u) = \frac{j_1(u)}{j_2(u)}$. Take two cutoffs $z_1 > z_0$. Then $b/z_1 < b/z_0$. We write $\frac{J_1(z_0)}{J_2(z_0)} - \frac{J_1(z_1)}{J_2(z_1)} = T_1 + T_2$,

where

$T_1 = \frac{\int_1^{b/z_1} r(u) j_2(u) f(z_0 u) du}{\int_1^{b/z_1} j_2(u) f(z_0 u) du} - \frac{\int_1^{b/z_1} r(u) j_2(u) f(z_1 u) du}{\int_1^{b/z_1} j_2(u) f(z_1 u) du}$

and

$T_2 = \frac{\int_1^{b/z_0} r(u) j_2(u) f(z_0 u) du}{\int_1^{b/z_0} j_2(u) f(z_0 u) du} - \frac{\int_1^{b/z_1} r(u) j_2(u) f(z_0 u) du}{\int_1^{b/z_1} j_2(u) f(z_0 u) du}.$

We show that both terms are non-negative.

First consider T_1. On the common interval $[1, b/z_1]$, define the density

$q(u; z) = \frac{j_2(u) f(zu)}{\int_1^{b/z_1} j_2(v) f(zv) dv}, \qquad 1 \le u \le b/z_1.$

Then T_1 is the difference between the expectation of the increasing function $r(u)$ under $q(u; z_0)$ and its expectation under $q(u; z_1)$. The likelihood ratio is

$\frac{q(u;z_1)}{q(u;z_0)} = C \frac{f(z_1 u)}{f(z_0 u)},$

where $C > 0$ is a constant independent of u. Differentiating the logarithm of this ratio with respect to $\log u$ gives, $\frac{d}{d \log u} \log \left(\frac{f(z_1 u)}{f(z_0 u)} \right) = \varepsilon_f(z_1 u) - \varepsilon_f(z_0 u)$. Since $z_1 u > z_0 u$ and ε_f is non-increasing, we have $\varepsilon_f(z_1 u) - \varepsilon_f(z_0 u) \le 0$.

Hence the distribution $q(u; z_1)$ is smaller than $q(u; z_0)$ in the first-order stochastic order. Since $r(u)$ is increasing, its expectation is lower under $q(u; z_1)$ than under $q(u; z_0)$. Therefore $T_1 \ge 0$.

Now consider T_2. This term compares two distributions with the same density kernel $j_2(u) f(z_0 u)$ but different upper bounds. The first distribution is defined on $[1, b/z_0]$, while the second is the same distribution truncated to the shorter interval $[1, b/z_1]$.

Let $D_0 = \int_1^{b/z_0} j_2(u) f(z_0 u) du, \qquad D_1 = \int_1^{b/z_1} j_2(u) f(z_0 u) du.$

Since $b/z_1 < b/z_0$ and the integrand is positive, $D_0 > D_1$. For $v \le b/z_1$, the two distribution functions are

$Q_0(v) = \frac{\int_1^v j_2(u) f(z_0 u) du}{D_0}$ for $1 \le v \le b/z_0$, and

$Q_1(v) = \frac{\int_1^v j_2(u) f(z_0 u) du}{D_1}$ for $1 \le v \le b/z_1$, $Q_1(v) = 1$, for $v > b/z_1$

Hence, $Q_0(v) - Q_1(v) = \int_1^v j_2(u) f(z_0 u) du \left(\frac{1}{D_0} - \frac{1}{D_1} \right) < 0$; for $1 \le v \le b/z_1$.

For $v \in (b/z_1, b/z_0]$, we have $Q_1(v) = 1$, while $Q_0(v) \le 1$. Hence $Q_0(v) \le Q_1(v)$ on the whole interval $[1, b/z_0]$. Therefore the distribution on the longer interval $[1, b/z_0]$ first-order stochastically dominates the distribution truncated to $[1, b/z_1]$.

Since $r(u)$ is increasing, its expectation under the longer-support distribution is at least as large as its expectation under the truncated distribution. Hence $T_2 \ge 0$.

Combining the two inequalities gives $\frac{J_1(z_0)}{J_2(z_0)} - \frac{J_1(z_1)}{J_2(z_1)} = T_1 + T_2 \ge 0$.Therefore, for any $z_1 > z_0$, $\frac{J_1(z_1)}{J_2(z_1)} \le \frac{J_1(z_0)}{J_2(z_0)}$. Thus $J_1(z^*)/J_2(z^*)$ is non-increasing in z^*.

Sketch of the converse

We briefly indicate why the converse holds; the full argument is given in Appendix A. Suppose that ε_f is not non-increasing. Then there exists an interval on which ε_f is strictly increasing. Choose two nearby cutoffs $z_0 < z_1$ and a number $c > 1$ such that the interval (z_0, cz_1) lies inside this region of increasing density elasticity. Hence, for every $u \in (1, c)$, both $z_0 u$ and $z_1 u$ lie in the region where ε_f is strictly increasing.

Choose weights that place most of the mass on this interval. Let $S > s > 0$ and define

$$j_2(u) = \begin{cases} S, & 1 < u < c, \\ \\ s, & u \geq c. \end{cases}$$

Set $j_1(u) = u j_2(u)$, so that $j_1(u)/j_2(u) = u$ is increasing.

For S/s sufficiently large, the ratio $J_1(z^*)/J_2(z^*)$ is governed by the behavior of the density on $(1, c)$. Since the density elasticity is increasing on the corresponding interval, the higher cutoff z_1 shifts mass toward larger values of u. Hence

$$\frac{J_1(z_0)}{J_2(z_0)} < \frac{J_1(z_1)}{J_2(z_1)}.$$

Thus the ratio is not non-increasing, contradicting the monotonicity property. Therefore the monotonicity property can hold for all admissible weights only if ε_f is non-increasing.

Remark 2. *The monotone selection effect studied in Section 2.1 is nested in Theorem 1. To see this, set $j_2(u) = 1$ and $j_1(u) = \phi(u)$, where ϕ is increasing. Then*

$$\frac{J_1(z^*)}{J_2(z^*)} = E_{z^*}[\phi(u)].$$

Hence Theorem 1 implies that $E_{z^*}[\phi(u)]$ is non-increasing in z^*. This is the same first-order stochastic dominance result obtained from the increasing generalized failure rate condition. The theorem therefore extends monotone selection from expectations of increasing functions to ratios of general weighted upper-tail aggregates.

The Pareto distribution provides the boundary case. With constant density elasticity, the distribution of relative types is invariant to the cutoff. Consequently, ratios of the form covered by Theorem 1 do not vary with the selection threshold. Non-Pareto tail behaviour is therefore what generates non-trivial comparative statics for aggregate ratios in models with endogenous selection.

Theorem 1 provides the link between the distributional primitives and the economic applications that follow. Once an aggregate object can be written as a ratio of weighted upper-tail integrals, its response to a higher threshold is determined by the relative weighting of high types and by the curvature of the underlying density. Log-log concavity gives a simple condition under which this response is monotone. The applications below use this result to study how endogenous selection affects aggregate productivity, trade elasticities, markups, concentration, default, and tail risk.

3.3 Application to variable markups

An important application of Theorem 1 arises in heterogeneous-firm models with variable markups. A convenient example is the translog demand system introduced by Bergin and Feenstra (2000), for which Rodríguez-López (2011) derives closed-form pricing rules. Let F denote the productivity distribution, and let $u = z/z^*$ denote productivity relative to the cutoff. Let $\rho(z)$ denote the real price set by a firm with productivity z. The sales share of such a firm is $s(z) := \frac{\rho(z)y(z)}{Y}$, where $y(z)$ is firm output and Y is aggregate output.

Under translog demand, the sales share depends on the firm's price relative to the price of the marginal firm,

$$s(z) = -\varsigma \ln\left(\frac{\rho(z)}{\rho(z^*)}\right).$$

Here ς is the preference parameter and $\rho(z^*)$ is the price charged by the marginal firm, whose sales share is zero.

With linear production, equilibrium prices are given by

$$\rho(z) = \Omega\left(e\frac{z}{z^*}\right)\frac{mc_t}{z},$$

where $\Omega(\cdot)$ denotes the principal branch of the Lambert Omega function, e is Euler's number and mc_t is marginal cost per unit of efficiency. If labor is the only input, then mc_t corresponds to the wage. The firm-level markup can therefore be written as a function of relative productivity

$$\mu(u) = \Omega(eu).$$

Since $\Omega(\cdot)$ is increasing on the positive real line, $\mu(u)$ is increasing in u.[7] Moreover, substituting

[7]Since $\Omega(x) < x$ for all $x > 0$, the weighted tail integral of markups is finite whenever the distribution has a finite first moment.

the pricing equation into the translog sales-share formula gives

$$s(u) = \varsigma[\Omega(eu) - 1],$$

so sales shares are also increasing in relative productivity.[8]

Aggregate markup is defined as total revenue divided by total variable cost. Since firm-level variable cost is equal to sales divided by the firm-level markup[9], aggregate markup can be written as

$$M(z^*) = \frac{\int_1^\infty s(u)\, dF_{z^*}(u)}{\int_1^\infty \frac{s(u)}{\mu(u)}\, dF_{z^*}(u)}.$$

This expression has exactly the ratio-of-integrals structure covered by Theorem 1. The numerator is total revenue, so take $j_1(u) = s(u)$, while the denominator is total variable cost, so take $j_2(u) = \frac{s(u)}{\mu(u)}$. The ratio of the weights is therefore $j_1(u)/j_2(u) = \mu(u)$, which is increasing in relative productivity u. Hence, under log-log concavity of the productivity density, Theorem 1 implies that the aggregate markup $M(z^*)$ is decreasing in the cutoff z^*.

This is an application of the main aggregation theorem, rather than a direct application of the generalized failure rate itself. In the earlier applications, the generalized failure rate entered directly into an equilibrium or optimality condition. Here, by contrast, the object of interest is an aggregate ratio: total revenue divided by total variable cost. Theorem 1 provides the link between the distributional condition and the economic aggregate. Log-log concavity orders the relative productivity distribution as the cutoff changes, and the theorem translates this ordering into monotonicity of the aggregate markup.

The economic mechanism is simple. When the cutoff rises, surviving firms are more productive in absolute terms. However, relative to the new cutoff, surviving firms become more concentrated near $u = 1$. Thus, among the firms that remain active, less mass is placed on firms far above the cutoff. Since firms farther above the cutoff charge higher markups, the aggregate markup places less weight on high-markup firms. Stronger selection therefore reduces the distributional component of aggregate markups.

This application illustrates the usefulness of Theorem 1. One does not need to solve the full equilibrium again in order to sign the effect of selection on this aggregate distortion. The model-specific equilibrium determines how the cutoff z^* responds to primitive shocks or policy changes. Theorem 1 determines how the distributional component of the aggregate markup responds to a movement in that cutoff.

[8]This is equation (7), p. 1143, of Rodríguez-López (2011).

[9]Since $\Omega(eu) > 1$, $s(u)/\mu(u) = \varsigma[1 - 1/\Omega(eu)] < 1$ and weighted tail integral of variable costs is finite.

An equivalent representation follows Edmond, Midrigan, and Xu (2023), who emphasize labor-weighted markups. Hence the aggregate markup can also be written as

$$M(z^*) = \frac{Y}{mc_t L} = \frac{\int_{z^*}^{\infty} \mu\left(\frac{z}{z^*}\right) l\left(\frac{z}{z^*}\right) dG(z)}{\int_{z^*}^{\infty} l\left(\frac{z}{z^*}\right) dG(z)}.$$

This is the same ratio-of-integrals structure, now with

$$j_1(u) = \mu(u)l(u), \qquad j_2(u) = l(u),$$

and therefore $j_1(u)/j_2(u) = \mu(u)$. Since $\mu(u)$ is increasing, Theorem 1 again implies that the labor-weighted aggregate markup is decreasing in the cutoff under log-log concavity.

The Pareto distribution provides the benchmark case. Under Pareto productivity, the distribution of relative productivity $u = z/z^*$ among active firms is invariant to the cutoff. As a result, the aggregate markup is independent of z^*, even though firm-level markups vary with relative productivity. This is the same scale-free property that makes Pareto selection models analytically tractable. For non-Pareto distributions, relative composition changes with the cutoff, and aggregate markups respond to selection.

These observations have direct macroeconomic implications. In models with a fixed number of firms, markups act as cost-push distortions and affect inflation dynamics through the Phillips curve. If a shock raises the productivity cutoff, the selection mechanism lowers the aggregate markup and therefore makes the economy more efficient. This effect is absent under Pareto tails, where the relative productivity distribution is invariant to the cutoff.

By contrast, in models with free entry, the decline in the aggregate markup has an additional general equilibrium effect. A lower markup reduces expected profits and suppresses entry, thereby dampening the expansionary effect of a positive TFP shock. In this sense, variable markups can smooth business cycle fluctuations induced by productivity shocks. This stabilizing channel is absent under Pareto productivity, where the aggregate markup is constant. The Pareto case may therefore generate excessive volatility and a stronger role for monetary policy. For a qualitative assessment, see Cooke and Damjanovic (2026).

3.4 Selection and the Herfindahl index

The Herfindahl-Hirschman index (HHI) summarizes the dispersion of market shares and therefore captures the composition of firms within an industry. Equilibrium outcomes often depend not only on the number of firms, but also on how market shares are distributed among them. As emphasized by Feenstra and Weinstein (2017), the HHI provides a tractable way to aggregate firm-level information.

This is important for our analysis because Theorem 1 implies that changes in the cutoff reshape the entire distribution of firm sizes. Stronger selection eliminates low-productivity firms and also changes the distribution of market shares among surviving firms. These compositional changes are precisely what the HHI captures.

Consider an economy with a mass N of heterogeneous firms. Firms differ in productivity z, and only firms with productivity above the endogenous cutoff z^* operate. Let $s(z/z^*)$ denote the sales share of an individual firm with productivity z.

Since total sales shares sum to one,

$$N \int_{z^*}^{\infty} s(z/z^*) f(z)\, dz = 1.$$

The Herfindahl-Hirschman index, defined as the sum of squared market shares, is therefore

$$HHI = N \int_{z^*}^{\infty} s\left(\frac{z}{z^*}\right)^2 f(z)\, dz = \frac{\int_{z^*}^{\infty} \left[s\left(z/z^*\right)\right]^2 f(z)\, dz}{\int_{z^*}^{\infty} s\left(z/z^*\right) f(z)\, dz}.$$

This expression has the ratio-of-integrals structure studied in Theorem 1. Hence, if the sales share function is increasing in relative productivity, Theorem 1 implies that, under log-log concavity of the productivity density, the Herfindahl index declines as the cutoff productivity z^* increases. Thus, stronger selection reduces equilibrium market concentration by compressing relative heterogeneity among active firms.

These relationships are consistent with existing theoretical results. Edmond, Midrigan, and Xu (2023) establish a monotonic relationship between markups and the Herfindahl index in a nested CES framework, while Cooke and Damjanovic (2026) show that economy-wide marginal costs are negatively related to the Herfindahl index in a translog environment.

3.5 Trade elasticity and the elasticity of substitution

We now revisit the discussion of the trade elasticity in Section 2.2.2. Using Proposition 1, we showed that the elasticity of export revenue with respect to both fixed and variable trade costs increases with the cutoff. We also discussed why the trade elasticity is independent of the elasticity of substitution across individual goods when productivity is unbounded Pareto, a result originally shown in Chaney (2008). Theorem 1 allows us to go further than this and show how the trade elasticity depends on the elasticity of substitution under general conditions.

Recall from equation (7) that the elasticity of aggregate export revenue with respect to variable trade costs is

$$-\frac{d\log R}{d\log\tau} = h_g\left(z^*\right) - \frac{d\log E_{z^*}u^{\sigma-1}}{d\log z^*}. \tag{15}$$

The only term in equation (15) that depends directly on σ is the cutoff elasticity of the truncated relative moment. The following proposition gives the required monotonicity result.

Proposition 7. *If f is log-log concave, then $\frac{d\log E_{z^*}u^{\sigma-1}}{d\log z^*}$ is decreasing in σ.*

Proof. Consider two exponents $\sigma_2 > \sigma_1 > 1$. By Theorem 1, since $u^{\sigma_2-1}/u^{\sigma_1-1} = u^{\sigma_2-\sigma_1}$ is increasing in u, the ratio $E_{z^*}u^{\sigma_2-1}/E_{z^*}u^{\sigma_1-1}$ is decreasing in z^*. Therefore,

$$\frac{d}{d\log z^*}\log\left(\frac{E_{z^*}u^{\sigma_2-1}}{E_{z^*}u^{\sigma_1-1}}\right) < 0.$$

Equivalently, $\frac{d\log E_{z^*}u^{\sigma_2-1}}{d\log z^*} < \frac{d\log E_{z^*}u^{\sigma_1-1}}{d\log z^*}$. Hence $\frac{d\log E_{z^*}u^{\sigma-1}}{d\log z^*}$ is decreasing in σ. □

Applying Proposition 7 shows that the second term in (15) decreases with σ. It follows that, for any given cutoff z^*, the trade elasticity with respect to variable trade costs increases with the elasticity of substitution σ. This is a result of independent interest. In the Armington model, the trade elasticity is determined by the elasticity of substitution between goods, and so the trade elasticity rises with the elasticity of substitution trivially. The same result holds in Krugman's (1980) analysis with endogenous product variety. Proposition 7 provides a formal re-statement of these ideas with firm heterogeneity and shows the same basic property holds across a wide class of models. Again, only in the Pareto case, when the trade elasticity is independent of the cut-off, is the same elasticity independent of the elasticity of substitution across goods.

4 Conclusion

This paper studies the role of distributional tail properties in economic environments with endogenous selection. We use the generalized failure rate and log-log concavity as analytical tools for characterizing how aggregate outcomes respond to changes in equilibrium thresholds. These objects provide a unified framework for analyzing selection margins in models with idiosyncratic characteristics.

We show that decreasing elasticity of the density implies monotonicity of relative tail distributions and yields general comparative statics results for ratios of weighted upper-tail integrals. This structure allows many aggregate economic variables, including average productivity, aggregate markups, market concentration, trade elasticities, and equilibrium default thresholds, to be analysed within a common framework. In particular, the results highlight how tighter selection changes relative heterogeneity among active agents and thereby affects aggregate distortions, selection effects, and risk-taking incentives.

The analysis also clarifies the special role of Pareto distributions. Under Pareto tails, the relative tail distribution is invariant to the selection threshold. This invariance eliminates composition effects and simplifies equilibrium responses. Departures from Pareto behaviour therefore generate richer comparative statics and provide a tractable channel through which distributional curvature influences aggregate outcomes.

Overall, the results suggest that log-log concavity is a useful primitive condition for studying endogenous selection in macroeconomic, financial, trade, and industrial organisation models. By linking threshold comparative statics to the behaviour of relative tail distributions, the paper provides a simple way to identify when selection matters for aggregate outcomes and when it is neutralised by the Pareto benchmark.

A Appendix: Proof of the Converse in Theorem 1

We prove the converse direction of Theorem 1 by contradiction. Suppose that the monotonicity property in Theorem 1 holds for all admissible positive weighting functions j_1 and j_2 such that $j_1(u)/j_2(u)$ is increasing. We show that ε_f must be non-increasing.

Suppose, to the contrary, that ε_f is not non-increasing. Then there exists a region of the support on which the density elasticity increases. In particular, choose an interval $[z_0, b_0]$ in the support on which ε_f is strictly increasing. Let

$$c = \left(\frac{b_0}{z_0}\right)^{1/2} > 1, \qquad z_1 = cz_0.$$

Then $cz_1 = c^2 z_0 = b_0$. Hence, for every $u \in (1, c)$, both $z_0 u$ and $z_1 u$ belong to (z_0, b_0), and $z_1 u > z_0 u$. Since ε_f is strictly increasing on this interval,

$$\varepsilon_f(z_1 u) > \varepsilon_f(z_0 u), \qquad u \in (1, c).$$

Define two probability densities on $(1, c)$,

$$p_k(u) = \frac{f(z_k u)}{\int_1^c f(z_k v)\, dv}, \qquad k = 0, 1.$$

Their likelihood ratio is

$$\frac{p_1(u)}{p_0(u)} = \frac{\int_1^c f(z_0 v)\,dv}{\int_1^c f(z_1 v)\,dv}\frac{f(z_1 u)}{f(z_0 u)}.$$

Taking the derivative with respect to $\log u$ gives

$$\frac{d}{d\log u}\log\left(\frac{p_1(u)}{p_0(u)}\right) = \varepsilon_f(z_1 u) - \varepsilon_f(z_0 u) > 0, \qquad u \in (1, c).$$

Thus $p_1(u)/p_0(u)$ is strictly increasing in u. Therefore p_1 dominates p_0 in the monotone likelihood ratio order, and hence also in the sense of first-order stochastic dominance. Since the function u is strictly increasing, it follows that $\int_1^c u\,p_1(u)\,du > \int_1^c u\,p_0(u)\,du$. Equivalently,

$$\frac{\int_1^c uf(z_1 u)\,du}{\int_1^c f(z_1 u)\,du} > \frac{\int_1^c uf(z_0 u)\,du}{\int_1^c f(z_0 u)\,du}.$$

After cross-multiplication, this inequality becomes $A > 0$, where

$$A = \int_1^c uf(z_1 u)\,du \times \int_1^c f(z_0 u)\,du - \int_1^c uf(z_0 u)\,du \times \int_1^c f(z_1 u)\,du \tag{16}$$

We now construct weighting functions that make the ratio of integrals arbitrarily close to the conditional expectations above. Let $S > s > 0$ and define

$$j_2(u) = \begin{cases} S, & 1 < u < c, \\ s, & u \geq c. \end{cases}$$

Set $j_1(u) = uj_2(u)$. Then $j_1(u)/j_2(u) = u$, which is increasing in u.

For any cutoff z, changing variables $x = zu$ gives

$$\frac{J_1(z)}{J_2(z)} = \frac{\int_1^{b/z} uj_2(u)f(zu)\,du}{\int_1^{b/z} j_2(u)f(zu)\,du}.$$

We will show that, for S/s sufficiently large,

$$\frac{J_1(z_1)}{J_2(z_1)} > \frac{J_1(z_0)}{J_2(z_0)}.$$

This contradicts the asserted monotonicity property, since $z_1 > z_0$ and $j_1(u)/j_2(u)$ is increasing.

The desired inequality is equivalent to

$$\frac{\int_1^{b/z_1} u\,j_2(u)\,f(z_1 u)\,du}{\int_1^{b/z_1} j_2(u)\,f(z_1 u)\,du} > \frac{\int_1^{b/z_0} u\,j_2(u)\,f(z_0 u)\,du}{\int_1^{b/z_0} j_2(u)\,f(z_0 u)\,du}.$$

Cross-multiplying positive denominators, this is equivalent to

$$\left(\int_1^{b/z_1} u\, j_2(u)\, f(z_1 u)\, du\right)\left(\int_1^{b/z_0} j_2(u)\, f(z_0 u)\, du\right)$$
$$-\left(\int_1^{b/z_0} u\, j_2(u)\, f(z_0 u)\, du\right)\left(\int_1^{b/z_1} j_2(u)\, f(z_1 u)\, du\right) > 0.$$

Split each integral at c. Since $cz_1 = b_0 < b$, we have $c < b/z_1$ and also $c < b/z_0$. Define

$$N_k^0 = \int_1^c uf(z_k u)\, du, \qquad D_k^0 = \int_1^c f(z_k u)\, du,$$

and

$$N_k^T = \int_c^{b/z_k} uf(z_k u)\, du, \qquad D_k^T = \int_c^{b/z_k} f(z_k u)\, du, \qquad k = 0, 1.$$

Then

$$\begin{aligned} \int_1^{b/z_k} uj_2(u) f(z_k u)\, du &= SN_k^0 + sN_k^T; \\ \int_1^{b/z_k} j_2(u) f(z_k u)\, du &= SD_k^0 + sD_k^T. \end{aligned}$$

Therefore the cross-multiplied difference is

$$\begin{aligned} &\left(SN_1^0 + sN_1^T\right)\left(SD_0^0 + sD_0^T\right) - \left(SN_0^0 + sN_0^T\right)\left(SD_1^0 + sD_1^T\right) \\ &= S^2\left(N_1^0 D_0^0 - N_0^0 D_1^0\right) \\ &\quad + sS\left(N_1^0 D_0^T + N_1^T D_0^0 - N_0^0 D_1^T - N_0^T D_1^0\right) \\ &\quad + s^2\left(N_1^T D_0^T - N_0^T D_1^T\right). \end{aligned}$$

The leading term is $S^2 A$, where $A = N_1^0 D_0^0 - N_0^0 D_1^0$ is the same as (16).

It remains only to control the tail terms. Let M be a finite common bound for the tail and interior integrals appearing above:

$$N_k^0,\ D_k^0,\ N_k^T,\ D_k^T \le M, \qquad k = 0, 1.$$

Then the absolute value of the mixed term is bounded by $4sSM^2$, and the absolute value of the final tail term is bounded by $2s^2M^2$. Hence the cross-multiplied expression is bounded below by

$$S^2 A - 4sSM^2 - 2s^2M^2.$$

Since $A > 0$, this lower bound is strictly positive for S/s sufficiently large.[10] Therefore,

$$\frac{J_1(z_1)}{J_2(z_1)} > \frac{J_1(z_0)}{J_2(z_0)}.$$

This contradicts the monotonicity property in Theorem 1, because $z_1 > z_0$ and $j_1(u)/j_2(u) = u$ is increasing. Thus the monotonicity property for all admissible weighting functions implies that ε_f must be non-increasing.

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[10] For example, a sufficient condition is $S/s > 20M^2/A$.

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